HSC Physics – Module 5: Advanced Mechanics – Motion in Gravitational Fields
Study Notes

- **Motion in Gravitational Fields**

Inquiry question: How does the force of gravity determine the motion of planets and satellites?

Newton's Law of Universal Gravitation

Newton's Law of Universal gravitation was based on several principles:

- Every object in the Universe attracts every other object with a gravitational force
- The gravitational force is directly proportional to the masses of the objects
- The gravitational force is inversely proportional to the square of the distance between them

Newton derived a relationship for the magnitude of the gravitational force acting between two objects:

$$\vec{F} = -\frac{GMm}{\vec{r}^2}$$

where:

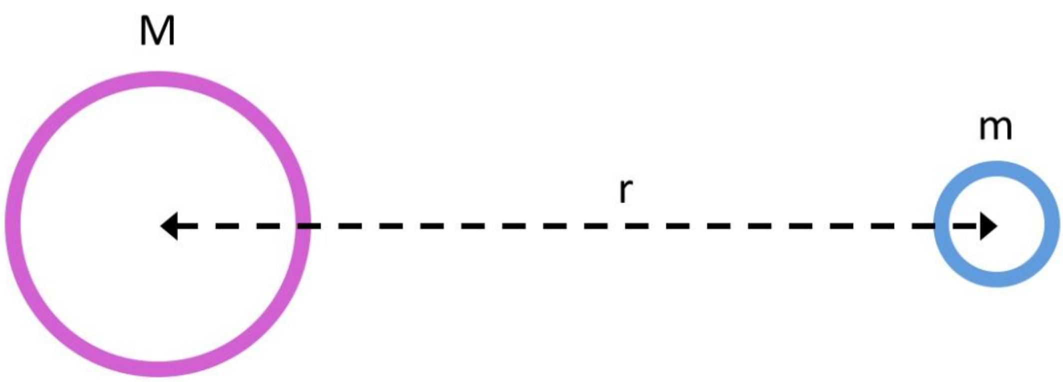
$\vec{F}$ is the gravitational force in N. (The negative sign in the equation indicates that the force is an attractive force)

G is the universal gravitation constant = $6.67 \times 10^{-11} Nm^2 kg^{-2}$

M is the mass of the larger body in kg

m is the mass of the smaller body in kg

r is the distance between the centres of the two masses in m, or, distance apart + the radii of each mass in m



The equation may also be seen or written as: $\vec{F} = \frac{GMm}{\vec{r}^2}$. Remember to include the direction as attractive.

Calculating gravitational field strength

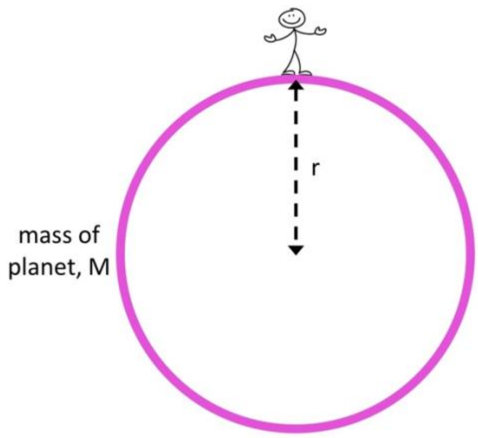
The strength of the gravitational field experienced by an object on or near the surface of any planet or moon is derived from the previous equation. If we assign a mass of 1kg to our object and also assume: $F_g = m_{object} \times g_{planet}$, we can conclude that:

$$mg = \frac{GMm}{\vec{r^2}}$$

and after cancelling, m, the smaller mass:

$$g = \frac{GM}{\vec{r^2}}$$

The unit for gravitational field strength is Nkg^{-1}



Calculating gravitational field strength at altitude

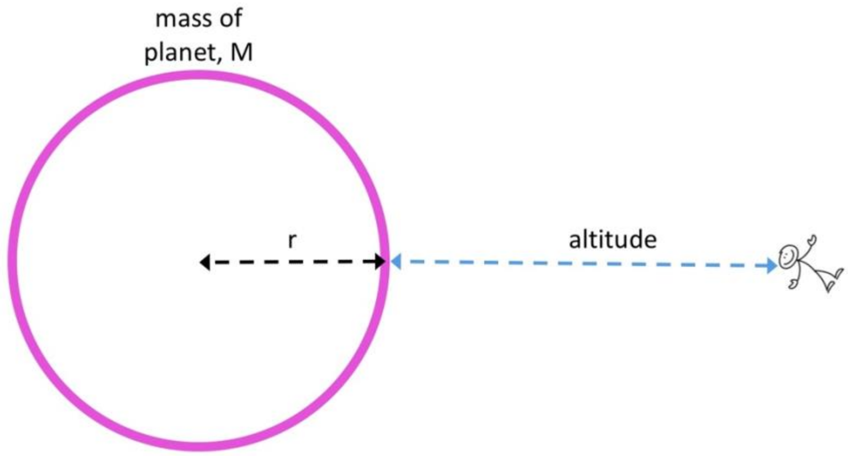
As an object moves from the surface of the Earth (or any planet) to a higher altitude, the strength of the gravitational field will decrease. These problems can be resolved by modifying the previous equation:

$$g = \frac{GM}{(R_E + altitude)^2}$$

where:

R_E is the radius of the Earth (or any planet)

$altitude$ is the distance that the object is from the surface of the Earth (or any other planet)

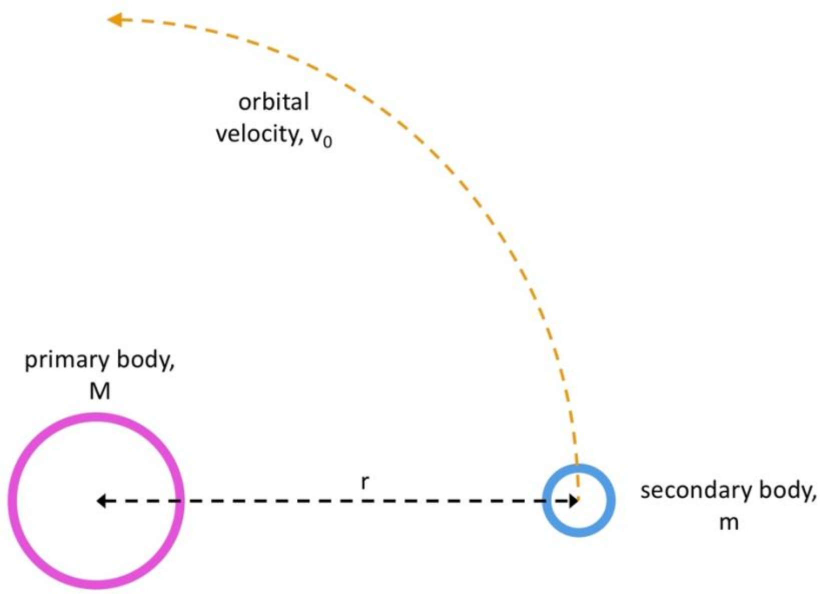


Useful constants: The following are given on data sheets and will be useful here:

- Mass of the Earth, $M_E = 6.0 \times 10^{24} kg$
- Radius of the Earth, $r_E = 6.371 \times 10^6 m$

Orbital Motion of Planets and Satellites

Newton's law of universal gravitation can be used to derive an equation for the average orbital speed, (v_o) of a planet (secondary body) around the Sun (primary body).



The gravitational force (F_g) between the Sun and a planet is the force that is responsible for the centripetal force (F_c) on the planet, keeping it in orbit around the Sun:

$$F_c = F_g$$

$$\frac{mv_o^2}{r} = \frac{GMm}{r^2}$$

where:

- m is the mass of the secondary body in kg
- v_o is the orbital velocity of the secondary body in m/s
- r is the distance between the centre of mass of the two bodies in m
- G is the universal gravitational constant = $6.67 \times 10^{-11} Nm^2kg^{-2}$
- M is the mass of the primary body in kg

dividing by *m*

$$\frac{v_o^2}{r} = \frac{GM}{r^2}$$

multiplying by *r*

$$v_o^2 = \frac{GM}{r}$$

taking the square root of each side

$$v_o = \sqrt{\frac{GM}{r}}$$

note: that the orbital speed does not depend on the mass of the satellite. It is proportional to the square root of the mass of the primary body and inversely proportional to the square root of the distance between the bodies.

The equation for orbital velocity can also be extended to **satellites in orbit around Earth.** Note that the radius, r, is the sum of the radius of the Earth and the altitude of the satellite. Thus, the equation becomes:

$$v_o = \sqrt{\frac{GM_E}{R_E + altitude}}$$

Period of Orbit
The period of orbit can be determined using:

$$T = \frac{2\pi r}{v_o}$$

Orbital Properties of Planets and Satellites

Types of Orbits

Spacecraft and satellites placed into orbit will generally be placed into one of two altitudes (orbits):

- Near Earth orbit (low Earth orbit)
- Geostationary and Geosynchronous orbit

Near Earth orbit:

Satellites in a near Earth orbit generally orbit the Earth at altitudes of between 250km and 1200km. Any lower than 250kms and atmospheric drag can impact the orbit of the satellite and any higher than 1200kms exposes the satellite to the Van Allen radiation belt. The Van Allen radiation belt is a region of high radiation trapped by the Earth's magnetic field and poses significant risk to space travellers and electronic equipment.

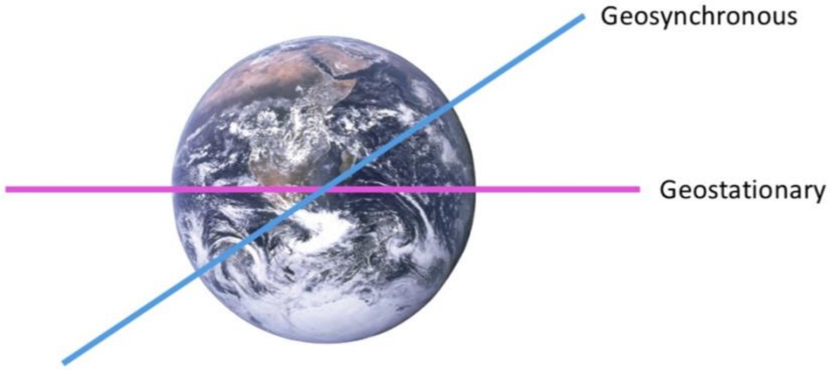
Most spacecraft or satellites in near Earth orbit exist between 250-400kms depending on their purpose. Spacecraft and satellites orbiting at 250km travel at approximately 27900 km/hr and take about 90 minutes to complete one orbit of the Earth. They can scan the total surface of the Earth every 24 hours.

Near Earth orbit satellites are used for a range of different purposes: These include mapping, monitoring weather and military surveillance. The International Space Station (ISS) is the largest near Earth satellite, orbiting at 400km in an almost circular orbit. The Hubble space telescope orbits at an altitude of 540km which is just above the Earth's atmosphere, allowing it to collect data that is not possible with Earth-based telescopes.

Geostationary and Geosynchronous Satellites:

A **geostationary satellite** stays over the same point on the Earth's surface above the equator. This requires the satellite to have an orbital period that is equal to that of the Earth – 1 day or 24 hours. Geostationary satellites orbit at altitudes of approximately 36,000km. Geostationary satellites are used for communications and global positioning systems (GPS).

A **geosynchronous satellite** will travel above any great circle. A great circle is any circle on the Earth's surface that has a radius which extends from the Earth's centre so that it has the same circumference as the equator. They have the same orbital properties of geostationary satellites.



Analysing satellite orbits:

When quantitatively analysing the orbital properties of satellites (period, orbital velocity, orbital radius and altitude), equations previously studied are used with the orbital properties discussed above:

$$v_o = \sqrt{\frac{GM}{r}}$$

$$T = \frac{2\pi r}{v_o}$$

A useful equation is derived from the equations above to determine the period independent of the orbital velocity:

$$\frac{2\pi r}{T} = \sqrt{\frac{GM}{r}}$$

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r}$$

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

Kepler's Laws of Planetary Motion

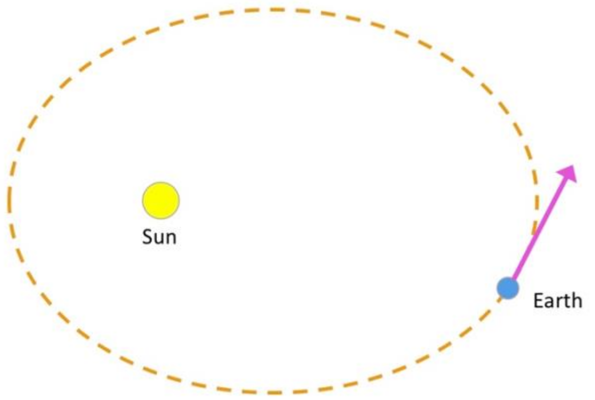
Before **Isaac Newton** published his work on The Law of Universal Gravitation, which provided a concise theory to explain the orbits of the planets around the Sun, **Johannes Kepler** had proposed a theory based on the work he inherited from **Tycho Brahe**. Tycho Brahe had made meticulous measurements and observations of the Sun, planets and stars over many years and Kepler was able to use this data to propose a model for the motion of the planets (Tycho Brahe did all his work before the invention of the telescope). Whilst Kepler's work was extensive, his theory is often summarised as three laws which although outdated by Newton's work, are still of historical significance and interest.

Kepler's Laws:

- First Law: The law of orbits
- Second Law: The law of areas
- Third Law: The law of periods

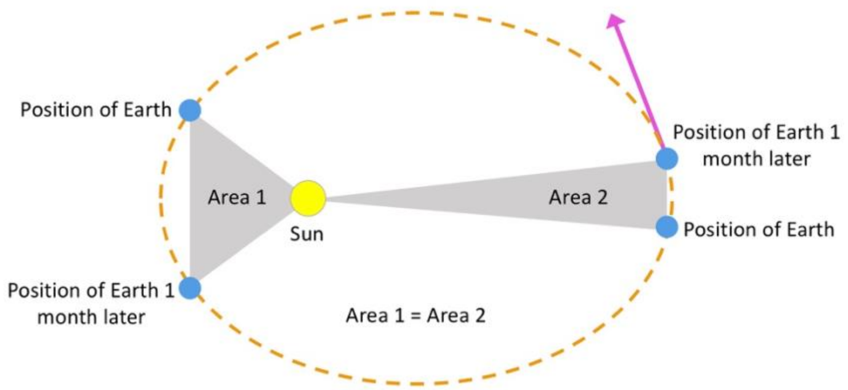
Kepler's first Law: The law of orbits

Kepler's first law states that all planets move in elliptical orbits with the sun at one focus and the other focus empty. The orbits of planets are almost circular and hard to distinguish on a scale diagram by eye.



Kepler's second Law: The law of areas

Kepler noticed from his analysis of the data that the speed of the planets changed as they orbited the Sun and that a line from the Earth to the Sun would move through different angles in equal periods of time. From these observations, Kepler discovered that the further a planet was from the Sun, the slower it would be moving. Further to this he concluded for his second law, that a line joining the planet to the Sun would sweep through equal areas in equal periods of time.



Kepler's third Law: The law of periods

Kepler was determined to find a relationship between the period of a planets orbit around the Sun and its average radius that was consistent for all planets. He eventually derived a relationship which is known as his third law: The cube of the average radius is proportional to the square of the orbital period of the planet.

This means that: $\frac{r^3}{T^2}$ is a constant for all planets.

Deriving from previous equations we find: $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$

And: $\frac{r^3}{T^2}$ (planet 1) $=$ $\frac{r^3}{T^2}$ (planet 2)

Force and Potential Energy in Gravitational Fields

Gravitational Potential Energy

The gravitational potential energy for objects near the surface of the Earth, $U = mgh$. In these situations, we say that $U = 0$ at ground level. As work is done to lift the object against the gravitational field, U increases as the height, h, increases. This is useful for objects which are close to the surface of the Earth but is not useful for interactions or situations where the gravitational field is not constant.

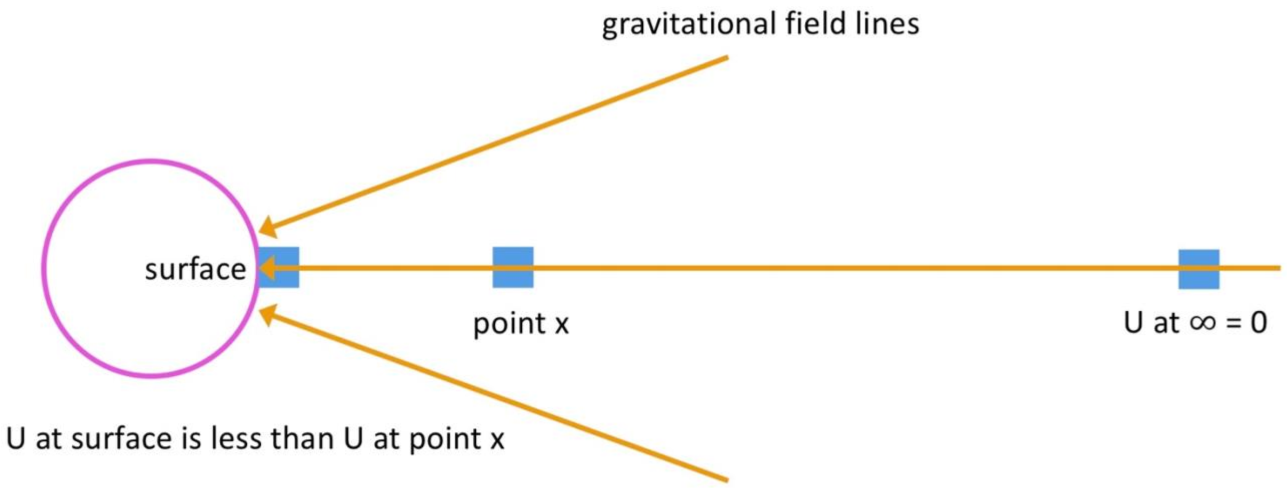
The equation for gravitational potential energy for any two objects with mass that are a distance, *r,* apart:

$$U = -\frac{GMm}{r}$$

Where:

- M is the mass of the planet in kg
- m is the mass of the object in kg
- r is the distance separating the masses in m

The equation is negative because according to the equation, at some point where $r = \infty$, $U = 0$. Now as r decreases (or as the objects become closer together), the object loses potential energy. If U decreases from 0, it must become negative! This key difference arises as for objects near the surface of the Earth, we assign ground level to be zero potential. For objects in space, we assign zero potential to be at a position very, very far away, that is, at ∞.



Derivation of Gravitational Potential Energy

Work is done to move an object away from a planet (work is done against the gravitational field). The work done on the object is equal to the change in potential energy of the object. If we are to derive the expression for gravitational potential energy, we need to equate work and gravitational potential energy in terms of *G, M, m* and *r*:

$$W = Fs$$

$$W = -\frac{GMm}{r^2} \times \Delta r$$

$$W = -\frac{GMm}{r}$$

$$U = -\frac{GMm}{r}$$

Escape Velocity

Isaac Newton conducted a thought experiment: If he was to throw a stone off the top of a tall building it will fall toward and strike the Earth. If he was to throw it harder, it would travel further before hitting the ground. Newton proposed that if he was able to throw it hard enough it would fall back toward the Earth as the Earth curved away, meaning that it would never actually hit the Earth.

With a similar analogy, we can consider throwing a stone straight up into the air. As we throw it harder it will travel faster. As we throw it faster and faster, it will travel higher and higher. If thrown fast enough it should rise up and continue to rise and slow down but never fall back to Earth. It will finally come to rest only when it has completely escaped the Earth's gravitational field. The initial velocity required to achieve this is known as **escape velocity.** This is the minimum speed that an object at the surface of the Earth will require if it is to escape into space and not be pulled back.

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Derivation of Escape Velocity

Escape velocity, v_{esc}, from a large body of mass M is achieved when the total orbital energy of an object is equal to 0, that is:

$$E_k + u = 0$$

$$\frac{1}{2}mv_{esc}^2 - \frac{GMm}{r} = 0$$

$$\frac{1}{2}mv_{esc}^2 = \frac{GMm}{r}$$

$$v_{esc}^2 = \frac{2GM}{r}$$

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

Energy in Orbits

Total Energy of a Planet or Satellite in its Orbit

An orbiting satellite has two important forms of energy: gravitational potential energy, *U*, and kinetic energy, *K*. The sum of these two energies give us what is known as the **total mechanical energy** of the system, *E*. The equation for the total mechanical energy of an orbiting body is derived by combining the centripetal force of a satellite being equal to the force of gravity, with the magnitude of the force from Newton's 2nd law and remembering the equation for the centripetal acceleration:

$$F = \frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$\therefore\ mv^2 = \frac{GMm}{r}$$

Incorporating the equation for kinetic energy by dividing by 2:

$$\frac{1}{2}mv^2 = \frac{GMm}{2r}$$

And combining this with the equation for the gravitational potential energy of a satellite, the total energy is:

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

Therefore, total mechanical energy of a system:

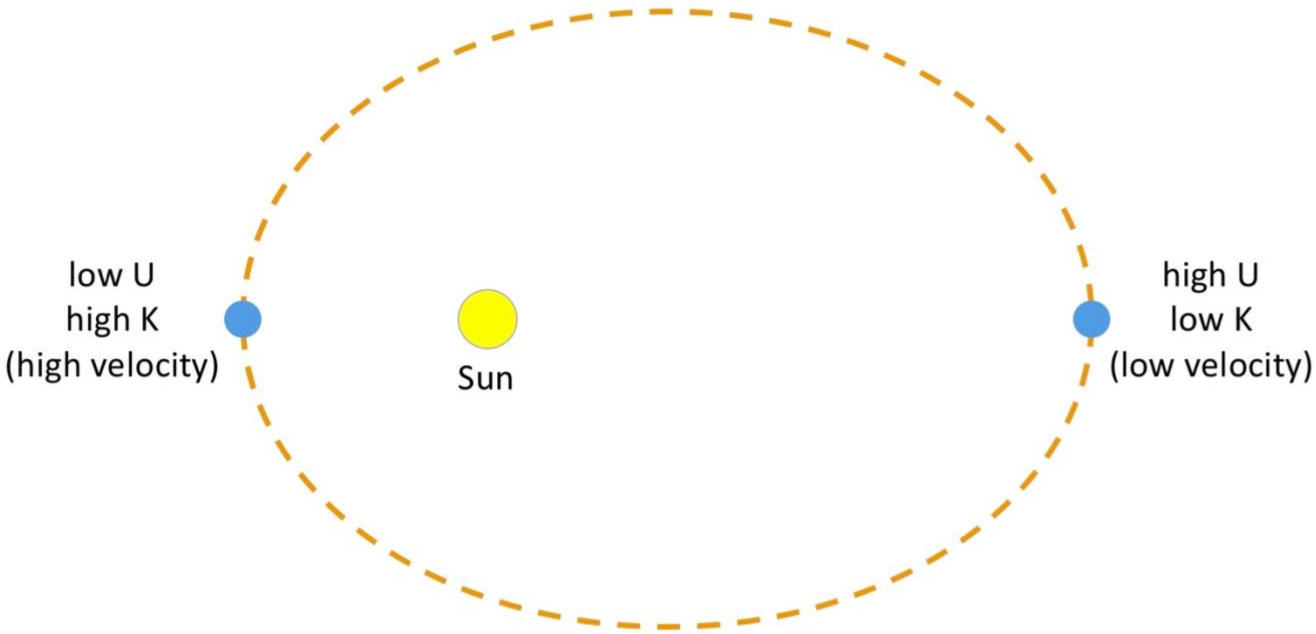
$$E = -\frac{GMm}{2r}$$

Considering Kepler's First Law

Remembering Kepler's first law: The planets move in elliptical orbits with the Sun at one focus. This means that the distance between the centre of masses, r, will vary as a planet orbits the Sun. This results in a change in the gravitational potential energy of the planet as r varies.

Importantly, the total mechanical energy of the system remains constant throughout its orbit. As a planet orbits in a position that is close to the Sun, its gravitational potential energy will decrease, however, the planets velocity will increase and also its kinetic energy. The decrease in gravitational potential energy is balanced by the increase in kinetic energy so that the total mechanical energy of the system is always a constant. In summary:

- Planets orbit the Sun in elliptical orbits
- Total mechanical energy is always constant
- As a planet orbits close to the Sun, the decrease in gravitational potential energy is balanced by the increase in kinetic energy
- As a planet orbits further away from the Sun, the increase in gravitational potential energy is balanced by the decrease in kinetic energy



Energy Changes That Occur When Satellites Move Between Orbits

When a satellite changes its orbit there will be a change in its gravitational potential energy. This is given by the formula:

$$\Delta U = U_f - U_i$$